e content for students of patliputra university

B. Sc. (Honrs) Part 2 paper 3

Subject:Mathematics

Title/Heading of topic: Cauchy Sequence, Cauchy general principles for convergence

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Cauchy Sequence

Definition 0.1. A sequence $\{x_n\}$ of real numbers is said to be ***Cauchy sequence*** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m > N \Rightarrow |x_n - x_m| < \varepsilon$.

A sequence is Cauchy if the terms eventually get arbitrarily close to each other.

Example 0.1. The sequence $\{\frac{1}{n}\}$ is Cauchy. To see this let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Now, if $n, m > N \Rightarrow |\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Example 0.2. The sequence $\{\frac{n}{n+1}\}$ is Cauchy. To see this let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Now, if $n, m > N \Rightarrow |\frac{n}{n+1} - \frac{m}{m+1}| = |\frac{m+1-n-1}{(n+1)(m+1)}| \leq |\frac{m-n}{nm}| < \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Lemma 1. *Let sequence $\{x_n\}$ be a Cauchy sequence of real numbers. Then $\{x_n\}$ is bounded.*

Proof. Since $\{x_n\}$ is a Cauchy sequence, then there exists $N \in \mathbb{N}$ such that if $n, m > N \Rightarrow |x_n - x_m| < 3$.

$$\text{if } n, m > N \Rightarrow |x_n - x_m| < 3$$

$$\text{let } m = N + 1, \text{ if } n > N \Rightarrow |x_n - x_{N+1}| < 3 \qquad \text{Note: } |x_n| - |x_{N+1}| \leq |x_n - x_{N+1}|$$

$$\Rightarrow |x_n| - |x_{N+1}| \leq |x_n - x_{N+1}| < 3$$

$$\text{if } n > N \Rightarrow |x_n| < 3 + |x_{N+1}|.$$

$$\text{Let } M = \max\{|x_1|, |x_2|, \cdots |x_N|, |x_{N+1}| + 3\}$$

$$\text{Now, if } n > N \Rightarrow |x_n| < 3 + |x_{N+1}| \leq M$$

$$\text{Now, if } n \leq N \Rightarrow |x_n| < \max\{|x_1|, |x_2|, \cdots |x_N|\} \leq M$$

$$\text{Thus } \forall\, n \in \mathbb{N},\ |x_n| \leq M.$$

□

Theorem 0.1. *[Cauchy Convergence Criterion]*

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a sequence of real numbers.

($\Rightarrow$) Suppose that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. We want to show that $\{x_n\}$ is Cauchy sequence.

Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to\infty} x_n = x \ \therefore \exists\, N \in \mathbb{N} \ni$

$$\text{if } n > N \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$$

$$\text{Also, if } m > N \Rightarrow |x_m - x| < \frac{\varepsilon}{2}.$$

$$\text{Now, if } n, m > N \Rightarrow |x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence .

($\Leftarrow$) Suppose that $\{x_n\}$ is a Cauchy sequence.We want to show that $\{x_n\}$ is convergent.

Let $\varepsilon > 0$ be given. Since $\{x_n\}$ is a Cauchy sequence, then by Lemma 1 it is bounded.

Hence $\{x_n\}$ has a converge subsequence $\{x_{n_k}\}$. Suppose $\lim\limits_{k\to\infty} x_{n_k} = x \in \mathbb{R}$.

There exist $N_1, N_2 \in \mathbb{N} \ni$ if $n, m > N_1 \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$

$$\text{and, if } k > N_2 \Rightarrow |x_{n_k} - x| < \frac{\varepsilon}{2}.$$

Now, fix $k > N_2$ such that $n_k > N_1$ and, if $n > N_1 \Rightarrow |x_n - x_{n_k}| < \frac{\varepsilon}{2}$ and $|x_{n_k} - x| < \frac{\varepsilon}{2}$.

Now, if $n > N_1 \Rightarrow |x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus $\{x_n\}$ converges . $\square$

Example 0.3. Prove that any sequence of real numbers $\{x_n\}$ which satisfies

$|x_n - x_{n+1}| = \frac{1}{5^n}, \qquad \forall\, n \in \mathbb{N}$ is convergent.

Proof.

$$
\begin{aligned}
\text{If } m > n \Rightarrow |x_n - x_m| &= |x_n - x_{n+1} + x_{n+1} + x_{n+2} + \ldots + x_{m-1} - x_m| \\
&\leq |x_n - x_{n+1}| + |x_{n+1} + x_{n+2}| + \ldots + |x_{m-1} - x_m| \\
&= \frac{1}{5^n} + \frac{1}{5^{n+1}} + \ldots + \frac{1}{5^{m-1}} \\
&= \frac{1}{5^{n-1}}\left(\frac{1}{5} + \frac{1}{5^2} + \ldots + \frac{1}{5^{m-n}}\right) \\
&= \frac{1}{5^{n-1}} \sum_{k=1}^{m-n} \frac{1}{5^k} \\
&= \frac{1}{5^{n-1}}\left(1 - \frac{1}{5^{m-n}}\right) \qquad \text{Note that: } \left(1 - \frac{1}{5^{m-n}}\right) < 1 \\
&< \frac{1}{5^{n-1}}.
\end{aligned}
$$

Let $\varepsilon > 0$ be given, choose $N \in \mathbb{N}$ such that $\dfrac{1}{5^{n-1}} < \varepsilon$.

$$\text{Now, if } n, m > N \Rightarrow |x_n - x_m| < \frac{1}{5^{n-1}} < \varepsilon.$$

Thus $\{x_n\}$ is convergent. □